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Causality Explains Why Spatial and Temporal Translations Commute: A Remark

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Under reasonable assumptions, it is proven that if a space-time has symmetries of translation type, then these symmetries form a commutative group.

Historically, the first physics-oriented nonplanar geometric models of space-time (proposed by Einstein) were smooth manifolds. However, it turned out that in the general case, physical space-times cannot be described by smooth manifolds: first, due to the equations of general relativity, they have *singularities* (Misner *et al.*, 1973), and second, due to quantum effects, space-times are locally nonsmooth. Therefore, a more general mathematical description of space-time is needed. A natural idea is to use for describing space-time a structure that is more physically fundamental than the structure of a smooth manifold. The most fundamental structure related to space-time is the structure of *causality*; therefore, Buseman (1967), Kronheimer and Penrose (1967), and Pimenov (1970) suggested to describe a space-time as an *ordered set* $(M, \leq)$, with $a \leq b$ meaning that an event a can causally influence an event b. In Newtonian physics, if two events a and b are simultaneous, they can influence each other; in other words, we have $a \leq b$ and $b \leq a$ and therefore $\leq$ is not an order. Since Einstein, however, it is believed that such instantaneous action is impossible. In view of that, we will assume that $\leq$ is an order.

Definition 1. By a *space-time*, we mean an ordered set $(M, \leq)$ that has at least one pair (a, b). A 1–1 mapping g: $M \to M$ is called a *symmetry* if it preserves causality, i.e., if for every a and b, $a \leq b$ iff $g(a) \leq g(b)$.

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Comments

- It is easy to see that symmetries form a group.
- Observers inside the space-time use numbers to describe the events. A function $x: M \to R$ that assigns numbers to events will be called a *coordinate*. Since we consider symmetric space-times (with symmetries generalizing translations in space-time), it is reasonable to consider only *inertial* coordinates, i.e., coordinates in which "translations" from the group G act as shifts $x \to x + \text{const}$. Of special interest are *temporal coordinates* t, i.e., coordinates in which if a can causally influence b, then $t(a) \leq t(b)$.

In special relativity, not only is it true that every time when $a \leq b$, we have $t(a) \leq t(b)$ for all temporal coordinates t, but the inverse is also true: the only reason why for some pairs of events a cannot influence b is that in some inertial coordinates, the time of b precedes the time of a. It is reasonable to make a similar assumption for our general case as well. Let us formulate it in mathematical terms.

Definition 2. Let $(M, \leq)$ be a space-time, and let G be a group of symmetries of M.

- By a *coordinate*, we understand a function $x: M \to R$.
- A coordinate x is called *inertial* if for every $g \in G$ there exists a number $s_x(g)$ such that $x(g(a)) = x(a) + s_x(g)$ for all a.
- A coordinate x is called *temporal* if $x(a) \leq x(b)$ whenever $a \leq b$.
- A pair (M, G) is called *natural* if the following condition holds: For every a, b, if $t(a) \leq t(b)$ for all inertial temporal coordinates t, then $a \leq b$.

Proposition. Let $(M, \leq)$ be a space-time, and let G be a group of symmetries of M. If (M, G) is a natural pair, then the group G is commutative (i.e., $g_1 g_2 = g_2 g_1$ for all $g_i \in G$).

Proof. 1. Let us first prove that for every inertial coordinate x and for every $g_1, g_2 \in G$, we have $s_x(g_1 g_2) = s_x(g_1) + s_x(g_2)$.

Indeed, for every x and for every $a \in M$, we have

$$x(g_1(g_2(a))) = x(g_2(a)) + s_x(g_1) = x(a) + s_x(g_2) + s_x(g_1)$$

On the other hand, $x(g_1(g_2(a))) = x(g_1 g_2(a)) = s_x(g_1 g_2) + x(a)$. Equating the resulting two expressions for $x(g_1(g_2(a)))$, we get the desired equality.

2. Let us finally prove that for every a, we have $g_1 g_2 = g_2 g_1$.

Indeed, for every a and for every temporal inertial coordinate t, we have $t(g_1 g_2(a)) = s_t(g_1) + s_t(g_2) + t(a)$ and similarly have $t(g_2 g_1(a)) = s_t(g_1) + s_t(g_2) + t(a)$. Hence, for every t, $t(g_1 g_2 a) = t(g_2 g_1 a)$ and therefore, $t(g_1 g_2 a) \geq t(g_2 g_1 a)$. Since the pair (M, G) is assumed to be natural, it follows that $g_1g_2(a) \geq g_2g_1(a)$. Similarly, $g_2g_1(a) \geq g_1g_2(a)$. Since $\leq$ is an order, we conclude that $g_1g_2(a) = g_2g_1(a)$ for all a, i.e., that $g_1g_2 = g_2g_1$. QED

Comment. This simple proof was influenced by the results presented (for topological groups) in Gładysz (1962, 1964) and Charin (1966, p. 139).

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